Optimal Boundary Control Method for a Flow Recirculation System

Nhan Nguyen*

Mark Ardema†

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Abstract

This paper is concerned with the optimal control of a class of distributed systems described by first order, quasilinear hyperbolic partial differential equations. In particular, this paper investigates an optimal control problem of a flow recirculation system; namely, a closed-circuit wind tunnel. The flow recirculation is modeled by the Euler equations with boundary conditions prescribing flow controls for the wind tunnel via a compressor performance model. The boundary control variables are further constrained by a set of ordinary differential equations representing dynamics of a lumped-parameter system that models a drive compressor speed regulation dynamics. Thus, the control variables of the lumped-parameter system influence the boundary control variables, which in turn influence the state variables of the distributed system. Necessary conditions for optimality are derived using variational principles. To illustrate the theory, we consider linear-quadratic optimal control for a linear hyperbolic partial differential equation with boundary control. Future work will apply these results to obtain a numerical solution of an optimal wind tunnel flow recirculation problem.

*Aerospace Engineer, NASA Ames Research Center, Moffett Field, CA 94035

†Professor, AIAA Associate Fellow, Department of Mechanical Engineering, Santa Clara University, Santa Clara, CA 95053

Introduction

The goal of this paper is to provide a general method for optimal control of a distributed system governed by first order, quasilinear hyperbolic partial differential equations coupled with a lumped-parameter system described by first order, nonlinear ordinary differential equations at the distributed system boundary.

For example, the control of a vibrating string may be described by the wave equation, written as a system of first order PDEs

$$\left\{\begin{matrix} y_{1,t} \\ y_{2,t} \end{matrix}\right\} + \begin{bmatrix} 0 & -c^2 \\ -1 & 0 \end{bmatrix} \left\{\begin{matrix} y_{1,x} \\ y_{2,x} \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\}$$

where $y_i\left(x,t\right)$, $i=1,2$ are the primary state variables denoting the time and spatial partial derivatives of the transverse displacement of the vibrating string. A control of the vibrating string may be defined on one of its boundary conditions such as

$$y_2\left(0,t\right) = u\left(t\right)$$

The boundary control variable $u\left(t\right)$ in turn may be controlled by a lumped-parameter system dynamics such as

$$\ddot{u}\left(t\right) + 2\zeta\omega_n\dot{u}\left(t\right) + \omega_n^2 u\left(t\right) = v\left(t\right)$$

where $v\left(t\right)$ is the secondary control variable.

One application of this class of problem is the optimal control of a flow recirculation in a wind tunnel. The majority of wind tunnels operated in the world are of a closed-circuit configuration. Air flow in these wind tunnels is recirculated through a closed-circuit duct by a compressor to achieve a desired air speed in a test section for testing a scaled model of a flight vehicle. One particular wind tunnel of this type is the NASA Ames 11-By 11-Foot Transonic Wind Tunnel (11-Ft TWT).

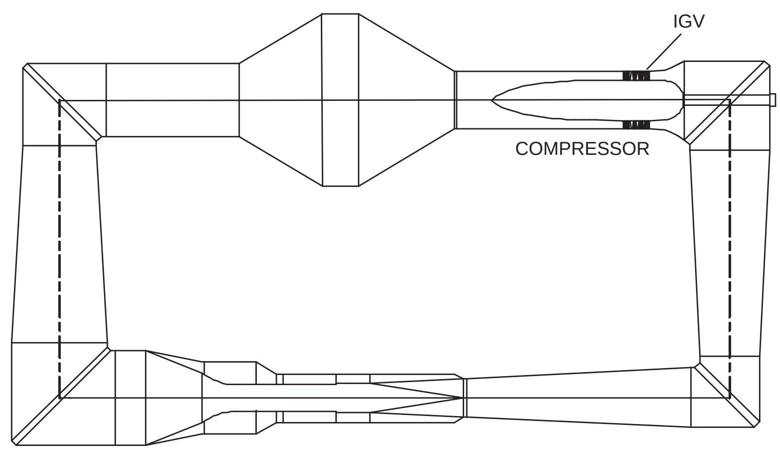

Fig. 1 - NASA Ames 11x11-Ft Transonic Wind Tunnel

With reference to Fig. 1, the NASA Ames 11-Ft TWT is a production wind tunnel capable of Mach 0.2 to Mach 1.5 at a variable stagnation pressure from 0.2 to 2.2 atmospheres. The air flow is delivered by a three-stage compressor driven by a set of four synchronous induction motors operated in tandem with a maximum power of 176 MW. After leaving the compressor, the air is decelerated through a wide angle diffuser and subsequently through an aftercooler to reduce to the stagnation temperature to within the wind tunnel operating limit. The air then passes through the back leg diffuser and then the settling chamber, where the air turbulence is reduced by a turbulence reduction system consisting of a set of screen and honeycomb flow conditioning devices. The conditioned air is then accelerated through a contraction and a flexible wall nozzle before entering the test section. The nozzle contour is designed to achieve a uniform Mach number distribution in the test section. The test section, measured 11 Ft x 11 Ft, provides the access to the scaled model, which is sting-mounted on a model support strut.

The test section aerodynamic conditions are normally controlled to ensure that the Mach number variation is within a prescribed tolerance. The Mach number control is accomplished by a combination of the compressor speed and the flap position of the variable-camber inlet guide vanes. There are two modes of Mach number control: command-following control and regulation control. In command-following control, the Mach number is transferred from one setpoint to another, whereas with regulation control, the Mach number is maintained at a setpoint during measurements of aerodynamic data of the scaled model.

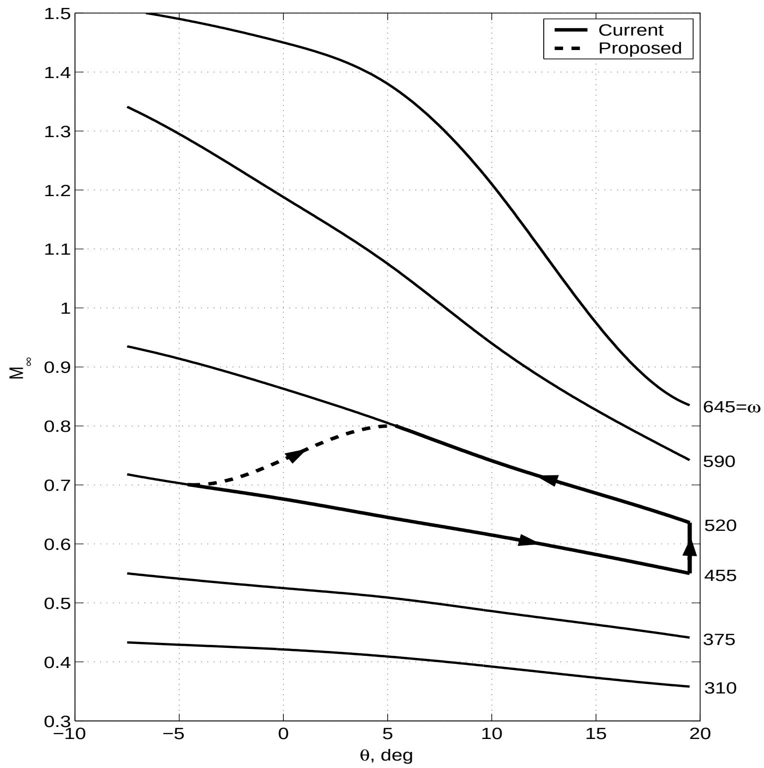

Fig. 2 - NASA Ames 11x11-Ft TWT Operating Envelope

With reference to Fig. 2, by varying the inlet guide vane (IGV) flap position θ at a given compressor speed ω, the Mach number can therefore be controlled. Presently, when a new Mach number setpoint is specified outside a current IGV range, the IGV have to be returned to the maximum flap position prior to changing the compressor speed setpoint. Additionally, the compressor speed must be controlled at each prescribed setpoint. An obvious improvement in the current Mach number control is to allow a simultaneous control of both the compressor speed and IGV flap deflection. Hence, this motivates the needs for examining trajectories of the control variables in the context of optimal control.

A further improvement is to consider a new mode of Mach number regulation control; namely, feedforward adaptive control. The current problem is that during a continuous pitch of the scaled model, the Mach number generally cannot be held close to the specified tolerance because of changes in the aerodynamics of the wind tunnel due to the changing model configuration in the test section. The idea is that if the aerodynamic disturbance could somehow be estimated via some learning algorithm such as neural network, then in theory the Mach number variation could be minimized by an optimal feedforward control such as linear-quadratic control. The idea of incorporating of neural network in the Mach number control is not new, since this concept has been investigated by Motter and others [1]. The use of optimal adaptive feedforward control in the context of this study, however, is considered novel and is proposed for a future work based on the distributed optimal control groundwork laid out in this paper.

A Wind Tunnel Distributed Model

In order to carry out a control investigation, a dynamic model of a wind tunnel is needed. Soeterboek [2] describes a method of modeling the test section Mach number by an experimental transfer function coupled with a time delay. In the past, the NASA Ames 11-Ft TWT Mach number control was developed using a quasi-steady state model based on the relationship between the Mach number and the experimentally derived compressor performance characteristics [3]. The use of a quasi-steady state model to describe the time varying behavior of the air flow thus circumvents the need to model the complex unsteady air flow. However, with the pseudo-steady state model, the time delay effect is not accurately captured and therefore can lead to poor control handling of disturbance rejection.

With the progress in computing technologies, modeling the unsteady air flow in a wind tunnel using fluid physics equations of motion for a control study has become a real possibility. This approach is adopted in this paper. Specifically, the unsteady air flow in a wind tunnel can be simulated by the 1-D Euler equations of motion with friction factors to model circuit losses. The use of a 1-D model for an actual 3-D flow in a wind tunnel is justified by the fact that for a control simulation only the spatially averaging effect of the flow is of a primary interest, rather than the detail features of the actual 3-D flow field. Furthermore, the 1-D model enables the use of experimentally determined friction factors to ensure that the steady state solution converges to the actual measured circuit pressure losses.

The three aerodynamic state variables of interest in a wind tunnel are the Mach number, M, the stagnation pressure, p_0, and stagnation temperature, T_0. It is noted that the mass flow rate, $\dot{m}$, through a wind tunnel, which is normally constant at a steady state operation, is a function of the three aerodynamic state variables. Thus, the mass flow rate can also be used in place of the Mach number as an aerodynamic state variable. The equations of motion derived for this set of state variables are

$$\mathbf{y}_t + \mathbf{A}\left(\mathbf{y}, x\right)\mathbf{y}_x + \mathbf{B}\left(\mathbf{y}, x\right) = \mathbf{0} \tag{1}$$

where $\mathbf{y} = \left\{\dot{m} \quad p_0 \quad T_0\right\}^T$ and

$$\mathbf{A} = \begin{bmatrix} u & \frac{pA}{p_0} & \frac{\dot{m}u}{2T_0} \\ \frac{\rho_0 c^2}{\rho A} & u\left[1-(k-1)\frac{T}{T_0}\right] & \frac{\rho_0 c^2 u}{T_0} \\ \frac{(k-1)T}{\rho A} & -\frac{2c^2}{kp_0 u}(k-1)(T_0-T) & u\left[1+(k-1)\frac{T}{T_0}\right] \end{bmatrix}$$

$$\mathbf{B} = \begin{Bmatrix} \frac{1}{2}\dot{m}u\frac{f}{D} \\ \frac{kp_0u^3}{2c^2}\frac{f}{D}\left[1-(k-1)\frac{T}{T_0}\right] \\ -u\frac{f}{D}(k-1)(T_0-T) \end{Bmatrix}$$

are expressed in terms of the usual aerodynamic variables: ρ_0, ρ, p, T, u, c; and the duct parameters: f, A, D; denoting respectively the stagnation and static density, static pressure and temperature, flow speed, speed of sound, friction factor, flow area, and hydraulic diameter. The specific heat ratio k is taken to be 1.4 for diatomic gases.

Equation (1) is the Euler equation with source terms for modeling the circuit loss behavior of a wind tunnel. This equation can also be written in the following conservation form

$$\mathbf{y}_t + \mathbf{f}\left(\mathbf{y}, x\right)_x + \mathbf{g}\left(\mathbf{y}, x\right) = \mathbf{0}$$

where $\mathbf{A} = \mathbf{f}_{\mathbf{y}}$ and $\mathbf{g} = \mathbf{B} - \mathbf{f}_x$.

Equation (1) is generally classified as a vector quasilinear hyperbolic partial differential equation since $\mathbf{A}$ and $\mathbf{B}$ are functions of $\mathbf{y}$, and additionally the eigenvalues of $\mathbf{A}$ are real and distinct. In fact, it can be shown that $\lambda\left(\mathbf{A}\right) = u, u \pm c$ are the eigenvalues, which have a physical interpretation as being the wave propagation speeds in a fluid medium.

The flow control problem of a wind tunnel is now posed as an initial-boundary value problem of Eq. (1). The initial condition specifies an initial steady state operation of the wind tunnel as follows

$$\mathbf{y}\left(x, 0\right) = \mathbf{h}\left(x\right) \tag{2}$$

The compressor is a point of discontinuity in the flow in a wind tunnel since work is done to raise the pressure across the compressor to compensate for the circuit loss. Therefore, the boundary condition is specified by the performance characteristics of the compressor. For convenience, let $x = 0$ and $x = L$ be defined as the compressor exit and inlet, respectively. By a dimensional analysis [4], the performance characteristics can generally be expressed as

$$\left\{ \begin{matrix} \frac{p_0(0,t)}{p_0(L,t)} & \frac{T_0(0,t)}{T_0(L,t)} - 1 \end{matrix} \right\} = f\left(\dot{m}\left(L, t\right), \omega, \theta\right) \tag{3}$$

By conservation of mass, it follows that

$$\dot{m}\left(0, t\right) = \dot{m}\left(L, t\right) \tag{4}$$

Thus, the boundary condition of Eq. (1) can generally be written as

$$\mathbf{g}\left(\mathbf{y}\left(0, t\right), \mathbf{y}\left(L, t\right), \mathbf{u}\right) = \mathbf{0} \tag{5}$$

where $\mathbf{u} = \left\{ \begin{matrix} \omega & \theta \end{matrix} \right\}^T$ is the primary compressor control vector.

Since the compressor speed is to be controlled at all times during a Mach number control, an auxiliary dynamic equation must be specified to relate the compressor speed to a secondary control variable. Physically, this dynamic equation describes the drive motors that control the compressor speed, which can be written as a first order ordinary differential equation

$$J_{motor}\dot{\omega} = T_{motor}\left(\omega, \omega_m\right) - T_{aero}\left(\mathbf{y}\left(0,t\right), \mathbf{y}\left(L,t\right)\right) \tag{6}$$

where ω_m is related to the drive motor control variable.

The optimal control problem is now formulated as a minimization of the following tracking-type cost functional that depends on both the distributed and lumped-parameter variables

$$J = \frac{1}{2}\int_0^T \int_0^L \left(\mathbf{y}-\mathbf{y}_d\right)^T \mathbf{P}\left(\mathbf{y}-\mathbf{y}_d\right) dxdt + \frac{1}{2}\int_0^T \left[q\left(\omega-\omega_d\right)^2 + r\theta^2 + s\omega_m^2\right] dt \tag{7}$$

An Optimal Control Problem

Optimal control of distributed systems modeled by partial differential equations with boundary controls has been widely studied (see, for example, Fursikov [5]). A distinguishing feature of the proposed model, however, is that one of the boundary control variables is in turn controlled by another separate process. This secondary control system in effect becomes a constraint to the boundary control variable of interest. Optimal control studies of combined distributed and lumped-parameter systems appears to be a new area of research interest.

In optimal control studies of distributed systems, a typical approach used by many investigators is to transform distributed systems into finite-dimensional systems by means of various numerical discretization techniques such as the finite-element method (see, for example, Becker [6]). In general, the finite-element method is a special case of weak-form solutions of partial differential equations. A weak-form solution implies that the partial differential equation can be recast into an integral form using weighting functions that satisfy certain smoothness and boundary value requirements.

Other direct approaches to studying the optimal control of partial differential equations have been examined by various workers. In one of his earlier works, Butkovskiy [7] discussed a maximum principle for first order, quasilinear partial differential equations. Kazemi [8] gave results of adjoint equations for a reduced system of first order partial differential equations, each with only one derivative of single independent variable. Hou and Yan applied an adjoint method to a weak form of a Navier-Stokes system [9].

The adjoint method based on calculus of variations enjoys a considerable popularity in the optimization studies of Euler and Navier-Stokes equations (see, for example, Jameson [10]). One of the features of the Euler equations is the presence of shocks where discontinuities in spatial derivatives occur. Variation of shock locations poses a considerable difficulty with the adjoint method as discussed by Cliff et al [11]. In the present study, the analysis is concerned with a subsonic flow for which the stagnation pressure is not discontinuous within the domain of solution and thus can be assumed to be piecewise smooth. For a transonic flow in a wind tunnel, the presence of a weak shock could be handled by a shock fitting technique whereby a shock location is prescribed a priori. The solution domain may then be divided into multiple subdomains where aerodynamic state variables upstream and downstream of the shock are related by the Rankine-Hugoniot relationship. The optimal control then admits a corner point where the state variables are discontinuous. Other approaches could also be formulated using a shock capturing technique which would necessitate the Euler equations be formulated in a conservation form and the aerodynamic state derivatives across the shock be computed by methods proposed by Ulrich [12] and others.

For the control problem at hand, let Ω be a compact subspace in $\mathbb{R}^2$ with a boundary Γ composed of $\Gamma_1 = [0, L]$ and $\Gamma_2 = [0, T]$. Consider a boundary control problem of a distributed system (S) governed by a system of first order, quasilinear partial differential equations as follows

$$D\mathbf{y} + \mathbf{B}\left(\mathbf{y}, \mathbf{w}, x\right) = \mathbf{0} \qquad \forall \left(x, t\right) \in \Omega \tag{8}$$

where D is a differential operator defined by

$$D\mathbf{y} = \mathbf{y}_t + \mathbf{A}(\mathbf{y}, x)\,\mathbf{y}_x$$

$\mathbf{y}(x,t) : \Omega \to \mathbb{R}^n$ in class C^1 is a distributed state vector, $\mathbf{w}(x,t) : \Omega \to \mathbb{R}^k$ belongs to a convex subset of admissible distributed control $\mathcal{W}_{ad} \subseteq \mathbb{R}^k$, $\mathbf{A}(\mathbf{y}, x) : \mathbb{R}^n \times \Gamma_1 \to \mathbb{R}^n \times \mathbb{R}^n$ is a square matrix of convective coefficients, and $\mathbf{B}(\mathbf{y}, \mathbf{w}, x) : \mathbb{R}^n \times \mathbb{R}^k \times \Gamma_1 \to \mathbb{R}^n$ is a source vector.

In addition, consider a lumped-parameter system (P) that possesses a certain control influence on the distributed system (S) as follows

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{y}(0,t), \mathbf{y}(L,t), \mathbf{u}, \mathbf{v}) \qquad \forall t \in \Gamma_2 \tag{9}$$

where $\mathbf{u}(t) : \Gamma_2 \to \mathbb{R}^m$ in class C^1 is a boundary control state vector, $\mathbf{v}(t) : \Gamma_2 \to \mathbb{R}^l$ belong to a convex subset of admissible lumped-parameter control $\mathcal{U}_{ad} \subseteq \mathbb{R}^l$, and $\mathbf{f}(\mathbf{y}(0,t), \mathbf{y}(L,t), \mathbf{u}, \mathbf{v}) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m$ is a vector function.

The coupling of the systems (S) and (P) enters in the boundary condition of Dirichlet type as follows

$$\mathbf{g}(\mathbf{y}(0,t), \mathbf{y}(L,t), \mathbf{u}) = \mathbf{0} \qquad \forall t \in \Gamma_2 \tag{10}$$

where $\mathbf{g}(\mathbf{y}(0,t), \mathbf{y}(L,t), \mathbf{u}) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a vector of boundary control functions.

For compactness, let $\mathbf{y}^0(t) = \mathbf{y}(0,t)$ and $\mathbf{y}^L(t) = \mathbf{y}(L,t)$ with the superscripts 0 and L henceforth meaning the values at the boundaries $x = 0$ and $x = L$.

To ensure well-posedness of boundary-value solutions of Eq. (8), the following initial conditions are specified

$$\mathbf{y}(x,0) = \mathbf{h}(x) \qquad \forall x \in \Gamma_1 \tag{11}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{12}$$

The following assumptions are required

A1: $\mathbf{w}(x,t)$ and $\mathbf{v}(t)$ are measurable, and squared-integrable in the Lebesgue sense with L^2 norms bounded on Ω.

A2: $\mathbf{A}(\mathbf{y},x)$ and $\mathbf{B}(\mathbf{y},\mathbf{w},x)$ are in at least class C^1 and satisfy the Lipschitz condition on Γ_1 and $\mathbb{R}^n \times \mathbb{R}^k \times \Gamma_1$, respectively, for some positive constants $C_i, i = 1, 2, ..., 4$ such that

$$|\mathbf{A}(\mathbf{y_2},x_2) - \mathbf{A}(\mathbf{y_1},x_1)| \leq C_1 |\mathbf{y}_2 - \mathbf{y}_1| + C_2 |x_2 - x_1|$$

$$|\mathbf{B}(\mathbf{y}_2,\mathbf{w}_2,x) - \mathbf{B}(\mathbf{y}_1,\mathbf{w}_1,x)| \leq C_3 |\mathbf{y}_2 - \mathbf{y}_1| + C_4 |\mathbf{w}_2 - \mathbf{w}_1|$$

A3: $\mathbf{g}\left(\mathbf{y}^0,\mathbf{y}^L,\mathbf{u}\right)$ and $\mathbf{h}(x)$ are continuous in at least class C^1, and $\mathbf{g}$ also satisfies the Lipschitz condition on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$

A4: $\mathbf{f}\left(\mathbf{y}^0,\mathbf{y}^L,\mathbf{u},\mathbf{v}\right)$ is continuous in at least class C^1 and satifies the Lipschitz condition on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l$.

Under the assumptions (A1)-(A3), Eq. (8) has a unique solution on Ω. The assumptions (A1) and (A4) also assert the existence and uniqueness of solutions to Eq. (9) on Γ_2.

Consider an optimal control problem of minimizing the following cost functional:

$$J(\mathbf{w},\mathbf{v}) = \int_0^T \int_0^L L_1(\mathbf{y},\mathbf{w},x)\,dxdt + \int_0^T L_2\left(\mathbf{y}^0,\mathbf{y}^L,\mathbf{u},\mathbf{v}\right)dt \tag{13}$$

subject to systems (S) and (P). For generality, the final time T is considered free.

Necessary Conditions for Optimal Control

Adopting the usual variational approach, a trial solution of Eq. (8) is proposed as a sum of the optimal solution $\mathbf{y}(x,t)$ and its variation

$$\mathbf{Y}(x,t) = \mathbf{y}(x,t) + \mathbf{z}(x,t)$$

where $\mathbf{z}(x,t) : \Omega \to \mathbb{R}^n$ in class C^1 is an admissible variation of $\mathbf{y}(x,t)$.

Similarly, a trial solution of the distributed control is formed as follows

$$\mathbf{W}(x,t) = \mathbf{w}(x,t) + \mathbf{r}(x,t)$$

where $\mathbf{w}(x,t)$ is the optimal distributed control and $\mathbf{r}(x,t) : \Omega \to \mathbb{R}^k$ in class C^1 is an admissible control variation of $\mathbf{w}(x,t)$.

A system of variational partial differential equations is obtained by taking the Fréchet differential of Eq. (8)

$$D\mathbf{z} + (\mathbf{A_y y}_x + \mathbf{B_y})\,\mathbf{z} + \mathbf{B_w r} = \mathbf{0}$$

where $\mathbf{A_y y}_x$, $\mathbf{B_y}$, and $\mathbf{B_w}$ are the Fréchet derivatives of the system (S).

Similarly, a trial solution of Eq. (9) is formed by letting

$$\mathbf{U}(t) = \mathbf{u}(t) + \mathbf{p}(t)$$

$$\mathbf{V}(t) = \mathbf{v}(t) + \mathbf{q}(t)$$

where $\mathbf{p}(t) : \Gamma_2 \to \mathbb{R}^m$ and $\mathbf{q}(t) : \Gamma_2 \to \mathbb{R}^l$ in class of C^1 are admissible variations of $\mathbf{u}(t)$ and $\mathbf{v}(t)$, respectively.

Then, the following system of variational ordinary differential equations is obtained

$$\dot{\mathbf{p}} = \mathbf{f}_{\mathbf{y}^0}\mathbf{z}^0 + \mathbf{f}_{\mathbf{y}^L}\mathbf{z}^L + \mathbf{f_u p} + \mathbf{f_v q}$$

The variational Dirichlet boundary condition, Eq. (10), for the system of variational partial differential equations is

$$\mathbf{g}_{\mathbf{y}^0}\mathbf{z}^0 + \mathbf{g}_{\mathbf{y}^L}\mathbf{z}^L + \mathbf{g_u p} = \mathbf{0}$$

Since the initial conditions of the systems (S) and (P) are given, their corresponding variations are required to vanish there. Hence

$$\mathbf{z}(x,0)=\mathbf{0}$$

$$\mathbf{p}(0)=\mathbf{0}$$

Applying the Lagrange multiplier yields the augmented cost functional as follows

$$J=\int_0^T\int_0^L\Big[L_1+\boldsymbol{\lambda}^T(D\mathbf{y}+\mathbf{B})\Big]dxdt+\int_0^T\Big[L_2+\boldsymbol{\mu}^T(-\dot{\mathbf{u}}+\mathbf{f})+\boldsymbol{\eta}^T\mathbf{g}\Big]dt$$

where $\boldsymbol{\lambda}(x,t)\,:\,\Omega\rightarrow\mathbb{R}^n$ is the distributed adjoint vector, $\boldsymbol{\mu}(t)\,:\,\Gamma_2\rightarrow\mathbb{R}^m$ is the lumped-parameter adjoint vector, $\boldsymbol{\eta}(t)\,:\,\Gamma_2\rightarrow\mathbb{R}^n$ is the boundary condition adjoint vector, and the superscript T denotes matrix transpose.

Now, introduce a dual Hamiltonian system for the systems (S) and (P) as follows

$$H_1(\mathbf{y},\mathbf{w},\boldsymbol{\lambda},x)=L_1+\boldsymbol{\lambda}^T\mathbf{B} \tag{14}$$

$$H_2\left(\mathbf{y}^0,\mathbf{y}^L,\mathbf{u},\mathbf{v},\boldsymbol{\mu},\boldsymbol{\eta}\right)=H_3+\boldsymbol{\eta}^T\mathbf{g} \tag{15}$$

where H_3 is the customary Hamiltonian function for the system (P) defined without the boundary condition constraint as follows

$$H_3\left(\mathbf{y}^0,\mathbf{y}^L,\mathbf{u},\mathbf{v},\boldsymbol{\mu}\right)=L_2+\boldsymbol{\mu}^T\mathbf{f} \tag{16}$$

Computing the Fréchet differential of the cost functional J yields

$$\begin{aligned}\delta J=&\int_0^T\int_0^L\Big[H_{1,\mathbf{y}}\mathbf{z}+H_{1,\mathbf{w}}\mathbf{r}+\boldsymbol{\lambda}^T(D\mathbf{z}+\mathbf{A}_{\mathbf{y}}\mathbf{y}_x\mathbf{z})\Big]dxdt+\int_0^L\left(H_1+\boldsymbol{\lambda}^TD\mathbf{y}\right)_{t=T}\delta t dx\\&+\int_0^T\left(H_{2,\mathbf{y}^0}\mathbf{z}^0+H_{2,\mathbf{y}^L}\mathbf{z}^L+H_{2,\mathbf{u}}\mathbf{p}+H_{2,\mathbf{v}}\mathbf{q}-\boldsymbol{\mu}^T\dot{\mathbf{p}}\right)dt+\left(H_2-\boldsymbol{\mu}^T\dot{\mathbf{u}}\right)_{t=T}\delta t\end{aligned}$$

Invoking the adjoint operation defined by an inner product over the domain of solution Ω to perform integration by parts as follows

$$\langle D\mathbf{z}, \boldsymbol{\lambda} \rangle = \langle \mathbf{z}, D^* \boldsymbol{\lambda} \rangle + b \tag{17}$$

where

$$\langle D\mathbf{z}, \boldsymbol{\lambda} \rangle = \iint_\Omega D\mathbf{z}^T \boldsymbol{\lambda} dx dt$$

D^* is the adjoint of the differential operator D, and b is the boundary condition term.

Then by means of the Green's Theorem, one gets

$$D^* \boldsymbol{\lambda} = -D^T \boldsymbol{\lambda} - \left[\left(\mathbf{A}_\mathbf{y} \mathbf{y}_x \right)^T \boldsymbol{\lambda} + \mathbf{A}_x^T \boldsymbol{\lambda} \right] \tag{18}$$

where D^T is a differential operator such that

$$D^T \boldsymbol{\lambda} = \boldsymbol{\lambda}_t + \mathbf{A}^T \boldsymbol{\lambda}_x$$

and the boundary term b

$$b = \oint_\Gamma \mathbf{z}^T \left(\mathbf{A}^T \boldsymbol{\lambda} dt - \boldsymbol{\lambda} dx \right) \tag{19}$$

Integrating the contour integral on the boundary Γ with vanishing initial variations yields

$$b = \oint_\Gamma \mathbf{z}^T \left(\mathbf{A}^T \boldsymbol{\lambda} dt - \boldsymbol{\lambda} dx \right) = \int_0^T \left[\left(\boldsymbol{\lambda}^L \right)^T \mathbf{A}^L \mathbf{z}^L - \left(\boldsymbol{\lambda}^0 \right)^T \mathbf{A}^0 \mathbf{z}^0 \right] dt + \int_0^L \boldsymbol{\lambda}^T \mathbf{z} \big|_{t=T} dx$$

This gives

$$\begin{aligned} \iint_\Omega \boldsymbol{\lambda}^T D\mathbf{z} dx dt = & \iint_\Omega \left(D^T \boldsymbol{\lambda}^T + \boldsymbol{\lambda}^T \mathbf{A}_\mathbf{y} \mathbf{y}_x + \boldsymbol{\lambda}^T \mathbf{A}_x \right) \mathbf{z} dx dt \\ & + \int_0^T \left[\left(\boldsymbol{\lambda}^L \right)^T \mathbf{A}^L \mathbf{z}^L - \left(\boldsymbol{\lambda}^0 \right)^T \mathbf{A}^0 \mathbf{z}^0 \right] dt + \int_0^L \boldsymbol{\lambda}^T \mathbf{z} \big|_{t=T} dx \end{aligned} \tag{20}$$

Substituting the foregoing results and integrating by parts in conjunction with vanishing variations at the initial time, the first variation of the cost functional J becomes

$$\begin{aligned}\delta J = \int_0^T \int_0^L & \left[\left(H_{1,\mathbf{y}} - D^T \boldsymbol{\lambda}^T - \boldsymbol{\lambda}^T \mathbf{A}_x\right)\mathbf{z} + H_{1,\mathbf{w}}\mathbf{r}\right] dx dt \\ & + \int_0^T \left[\left(\boldsymbol{\lambda}^L\right)^T \mathbf{A}^L \mathbf{z}^L - \left(\boldsymbol{\lambda}^0\right)^T \mathbf{A}^0 \mathbf{z}^0\right] dt + \int_0^L \left[H_1 \delta t + \boldsymbol{\lambda}^T d\mathbf{y}\right]_{t=T} dx \\ & + \int_0^T \left[H_{2,\mathbf{y}^0}\mathbf{z}^0 + H_{2,\mathbf{y}^L}\mathbf{z}^L + \left(H_{2,\mathbf{u}} + \dot{\boldsymbol{\mu}}^T\right)\mathbf{p} + H_{2,\mathbf{v}}\mathbf{q}\right] dt + \left[H_2 \delta t - \boldsymbol{\mu}^T d\mathbf{u}\right]_{t=T} \end{aligned} \tag{21}$$

where

$$d\mathbf{y}\,(x,T) = \mathbf{z}\,(x,T) + D\mathbf{y}\,(x,T)\delta t$$

$$d\mathbf{u}\,(T) = \mathbf{p}\,(T) + \dot{\mathbf{u}}\,(T)\,\delta t$$

The necessary conditions for minimizing the cost functional J may now be derived by requiring that each variational term in the first variation of the cost functional be zero for any arbitrary variation. Thus, the following associated distributed adjoint system (Σ) and lumped-parameter adjoint system (Π) are obtained

$$D^T \boldsymbol{\lambda} + \mathbf{A}_x^T \boldsymbol{\lambda} - H_{1,\mathbf{y}}^T = \mathbf{0} \tag{22}$$

$$\dot{\boldsymbol{\mu}} + H_{2,\mathbf{u}}^T = \mathbf{0} \tag{23}$$

Equation (22) is a system of first order, quasilinear hyperbolic partial differential equations in terms of the adjoint vector $\boldsymbol{\lambda}\,(x,t)$. Since the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{A}^T$ are the same, the adjoint system (Σ) preserves the characteristics of the original system (S).

In addition, two auxiliary algebraic equations are obtained as follows

$$\left(\mathbf{A}^0\right)^T \boldsymbol{\lambda}^0 - H_{2,\mathbf{y^0}}^T = \mathbf{0} \tag{24}$$

$$\left(\mathbf{A}^L\right)^T \boldsymbol{\lambda}^L + H_{2,\mathbf{y}^L}^T = \mathbf{0} \tag{25}$$

Solving for the adjoint vector $\boldsymbol{\eta}(t)$ from Eqs. (15) and (24) in terms of the Hamiltonian function H_3, assuming $\mathbf{y}(0,t)$ explicitly appears in the boundary condition $\mathbf{g}$, yields

$$\boldsymbol{\eta} = \left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1}\left[\left(\mathbf{A}^0\right)^T \boldsymbol{\lambda}^0 - H_{3,\mathbf{y^0}}^T\right]$$

Upon substitution, the following boundary condition for the adjoint system (Σ) is obtained

$$\left(\mathbf{A}^L\right)^T \boldsymbol{\lambda}^L + H_{3,\mathbf{y}^L}^T + \mathbf{g}_{\mathbf{y}^L}^T\left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1}\left[\left(\mathbf{A}^0\right)^T \boldsymbol{\lambda}^0 - H_{3,\mathbf{y^0}}^T\right] = \mathbf{0} \tag{26}$$

From the foregoing results, the adjoint system (Π) in terms of the Hamiltonian function H_3 may be expressed as follows

$$\dot{\boldsymbol{\mu}} + H_{3,\mathbf{u}}^T + \mathbf{g}_{\mathbf{u}}^T\left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1}\left[\left(\mathbf{A}^0\right)^T \boldsymbol{\lambda}^0 - H_{3,\mathbf{y^0}}^T\right] = \mathbf{0} \tag{27}$$

The necessary condition for an unbounded optimal control of the system (S) is obtained from

$$H_{1,\mathbf{w}} = \mathbf{0} \tag{28}$$

There are two special cases to be considered. The first case is when $\mathbf{w} = \mathbf{w}(t)$, for which the necessary condition gives

$$\int_0^L H_{1,\mathbf{w}} dx = \mathbf{0}$$

For the case when $\mathbf{w} = \mathbf{w}(x)$, the optimal control is obtained from

$$\int_0^T H_{1,\mathbf{w}} dt = \mathbf{0}$$

Similarly, the necessary condition for an unbounded optimal control of the system (P) is given by

$$H_{2,\mathbf{v}} = \mathbf{0} \tag{29}$$

The first variation also gives rise to the following transversality conditions

$$\int_0^L \boldsymbol{\lambda}^T(x,T)\,d\mathbf{y}(x,T)\,dx = 0 \tag{30}$$

$$\boldsymbol{\mu}^T(T)\,d\boldsymbol{u}(T) = 0 \tag{31}$$

Actually, a stronger transversality condition for the adjoint system (Σ) would be

$$\boldsymbol{\lambda}^T(x,T)\,d\mathbf{y}(x,T) = 0$$

These transversality conditions specify the final values of the adjoint vectors $\boldsymbol{\lambda}(x,t)$ and $\boldsymbol{\mu}(t)$. The known values of the adjoint vectors at the final time with the specified initial values of the state vectors establish a two-point boundary value problem.

In addition to the foregoing necessary conditions, two auxiliary final-time conditions are to be satisfied

$$\int_0^L H_1(x,T)\,dx = 0 \tag{32}$$

$$H_2(T) = 0 \tag{33}$$

A stronger condition would be

$$H_1(x,T) = 0$$

If the coefficient matrix $\mathbf{A}$ and the Hamiltonian function H_1 are not explicit functions of x and t, then the Hamiltonian function H_1 is constant along the characteristic direction. To show this, the characteristic method is used to transform the partial differential equations into ordinary differential equations by characteristic vectors $\mathbf{s} = \boldsymbol{\Phi}^{-1}\mathbf{y}$ and $\boldsymbol{\phi} = \boldsymbol{\Phi}^T\boldsymbol{\lambda}$ where $\boldsymbol{\Phi}$ is a matrix of the corresponding column eigenvectors of $\mathbf{A}$ such that $\mathbf{A} = \boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}$ and $\boldsymbol{\Lambda}$ is a diagonal matrix of the eigenvalues.

Then Eqs. (8) and (25) become

$$\frac{d\mathbf{s}}{dt} = -\boldsymbol{\Phi}^{-1}\mathbf{B}$$

$$\frac{d\boldsymbol{\phi}}{dt} = \boldsymbol{\Phi}^T H_{1,\mathbf{y}}^T$$

with the total derivative understood to be along each characteristic direction as defined by $\left(\frac{dx}{dt}\right)_i = \lambda_i\left(\mathbf{A}\right)$

Since the Hamiltonian function H_1 is not an explicit function of x and t, its time derivative can be computed using chain rule differentiation as follows

$$\frac{dH_1\left(\mathbf{y}, \mathbf{w}, \boldsymbol{\lambda}\right)}{dt} = H_{1,\mathbf{y}}\frac{d\mathbf{y}}{d\mathbf{s}}\frac{d\mathbf{s}}{dt} + H_{1,\boldsymbol{\lambda}}\frac{d\boldsymbol{\lambda}}{d\boldsymbol{\phi}}\frac{d\boldsymbol{\phi}}{dt} + H_{1,\mathbf{w}}\left(\mathbf{w}_t + \mathbf{w}_x\frac{dx}{dt}\right)$$

By the virtue of the necessary condition of Eq. (28), the foregoing expression can be simplified as

$$\frac{dH_1}{dt} = -H_{1,\mathbf{y}}\mathbf{B} + \mathbf{B}^T H_{1,\mathbf{y}}^T = 0 \tag{34}$$

Thus, the Hamiltonian function H_1 is constant along each characteristic direction. This result indicates that some relationships in optimal control for a system of ordinary differential equations may be also applicable for a system of first order hyperbolic partial differential equations through the use of the characteristic transformation.

It can be shown that the system (S) is asymptotically stable under a certain condition. Asymptotic stability provides a certain measure of controllability, which enables the system (S) to transfer from some initial target set to the origin in a finite time. Existence of an optimal control solution is then asserted.

The stability of the system (S) may be examined in the context of the Lyapunov's direct method in conjunction with the characteristic method by forming a positive-definite continuous function $V\left(\mathbf{y}\right) > 0$. The time derivative of $V\left(\mathbf{y}\right)$ is computed by chain rule differentiation

$$\dot{V}\left(\mathbf{y}\right) = \frac{dV}{d\mathbf{y}}\frac{d\mathbf{y}}{d\mathbf{s}}\frac{d\mathbf{s}}{dt}$$

The Lyapunov's direct method yields

$$\dot{V}(\mathbf{y}) = \frac{dV}{d\mathbf{y}}\mathbf{\Phi}\left(-\mathbf{\Phi}^{-1}\mathbf{B}\right) = -\frac{dV}{d\mathbf{y}}\mathbf{B} < 0$$

Let $V(\mathbf{y}) = \mathbf{y}^T\mathbf{y}$, then the asymptotic stability for the system (S) requires that $2\mathbf{y}^T\mathbf{B} > \mathbf{0}$. This implies

$$\|\mathbf{y} - \mathbf{B}\| < \sqrt{\|\mathbf{y}\|^2 + \|\mathbf{B}\|^2} \tag{35}$$

By a similar analysis, it can be shown that the adjoint system (Σ) possesses the following stability requirement

$$\|\boldsymbol{\lambda} - \left(\mathbf{A}^T\boldsymbol{\lambda} - \mathbf{H}_{1,\mathbf{y}}^T\right)\| < \sqrt{\|\boldsymbol{\lambda}\|^2 + \|\mathbf{A}^T\boldsymbol{\lambda} - \mathbf{H}_{1,\mathbf{y}}^T}\|^2 \tag{36}$$

Now, suppose that the boundary control state vector $\mathbf{u}(t)$ is partially constrained so that $\mathbf{u} = \begin{Bmatrix}\mathbf{U} & \mathbf{V}\end{Bmatrix}^T$ where $\mathbf{U}(t) : \Gamma_2 \to \mathbb{R}^N, N < m$ is a constrained boundary state vector and $\mathbf{V}(t) : \Gamma_2 \to \mathbb{R}^{m-N}$ is a boundary control vector. The constraint on $\mathbf{U}$ is defined by the following lumped-parameter system (Q)

$$\dot{\mathbf{U}} = \mathbf{f}\left(\mathbf{y}(0,t), \mathbf{y}(L,t), \mathbf{U}, \mathbf{v}\right) \tag{37}$$

Then, the optimality of the systems (S) and (Q) is defined by the following partial differential equation

$$\boldsymbol{\lambda}_t + \mathbf{A}^T\boldsymbol{\lambda}_x + \left(\mathbf{A}_x^T - \mathbf{B}_{\mathbf{y}}^T\right)\boldsymbol{\lambda} - L_{1,\mathbf{y}}^T = \mathbf{0} \tag{38}$$

subject to the boundary condition

$$\left[\mathbf{f}_{\mathbf{y}^L}^T - \mathbf{g}_{\mathbf{y}^L}^T\left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1}\mathbf{f}_{\mathbf{y}^0}^T\right]\boldsymbol{\mu} + \mathbf{g}_{\mathbf{y}^L}^T\left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1}\left(\mathbf{A}^0\right)^T\boldsymbol{\lambda}^0 + \left(\mathbf{A}^L\right)^T\boldsymbol{\lambda}^L + \left[L_{2,\mathbf{y}^L}^T - \mathbf{g}_{\mathbf{y}^L}^T\left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1}L_{2,\mathbf{y}^0}^T\right] = \mathbf{0} \tag{39}$$

and the following ordinary differential equation

$$\dot{\boldsymbol{\mu}} + \left[\mathbf{f}_{\mathbf{U}}^T - \mathbf{g}_{\mathbf{U}}^T\left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1}\mathbf{f}_{\mathbf{y}^0}^T\right]\boldsymbol{\mu} + \mathbf{g}_{\mathbf{U}}^T\left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1}\left(\mathbf{A}^0\right)^T\boldsymbol{\lambda}^0 + \left[L_{2,\mathbf{U}}^T - g_{\mathbf{U}}^T\left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1}L_{2,\mathbf{y}^0}^T\right] = \mathbf{0} \tag{40}$$

with the following optimal controls

$$L_{1,\mathbf{w}}^T + \mathbf{B}_{\mathbf{w}}^T \boldsymbol{\lambda} = \mathbf{0} \tag{41}$$

$$L_{2,\mathbf{v}}^T + \mathbf{f}_{\mathbf{v}}^T \boldsymbol{\mu} = \mathbf{0} \tag{42}$$

$$L_{2,\mathbf{V}}^T - g_{\mathbf{V}}^T \left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1} L_{2,\mathbf{y}^0}^T + \mathbf{g}_{\mathbf{V}}^T \left(\mathbf{g}_{\mathbf{y}^0}^T\right)^{-1} \left(\mathbf{A}^0\right)^T \boldsymbol{\lambda}^0 = \mathbf{0} \tag{43}$$

If the controls are bounded by inequality constraints, then the necessary conditions may be obtained by the minimum principle. To illustrate the minimum principle, let the candidate optimal state and control vectors be denoted with the superscript *. Then, if the optimal control vectors $\mathbf{w}^*$ and $\mathbf{v}^*$ were to be perturbed by some measure, the cost functional J with the perturbed controls must be greater than that with the optimal controls. That is

$$J\left(\mathbf{w}^*, \mathbf{v}^*\right) < J\left(\mathbf{w}^* + \mathbf{r}, \mathbf{v}^* + \mathbf{q}\right)$$

This implies that

$$\int_0^T \int_0^L H_1\left(\mathbf{y}^*, \mathbf{w}^*, \boldsymbol{\lambda}, x\right) dx dt < \int_0^T \int_0^L H_1\left(\mathbf{y}^*, \mathbf{w}^* + \mathbf{r}, \boldsymbol{\lambda}, x\right) dx dt$$

$$\int_0^T H_2\left(\mathbf{y}^{0*}, \mathbf{y}^{L*}, \mathbf{u}^*, \mathbf{v}^*, \boldsymbol{\mu}\right) dt < \int_0^T H_2\left(\mathbf{y}^{0*}, \mathbf{y}^{L*}, \mathbf{u}^*, \mathbf{v}^* + \mathbf{q}, \boldsymbol{\mu}\right) dt$$

Thus, the above inequalities indicate that the optimal controls $\mathbf{w}^*$ and $\mathbf{v}^*$ are those that minimize the Hamiltonian functions H_1 and H_2, respectively. This statement equivalently leads to the following minimum principle

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left[H_1\left(\mathbf{y}, \mathbf{w}, \boldsymbol{\lambda}, x\right)\right] \tag{44}$$

$$\mathbf{v}^* = \arg\min_{\mathbf{v}} \left[H_2\left(\mathbf{y}^0, \mathbf{y}^L, \mathbf{u}, \mathbf{v}, \boldsymbol{\mu}\right)\right] \tag{45}$$

If $\mathbf{w}=\mathbf{w}\left(t\right)$, then Eq. (44) is replaced by

$$\mathbf{w}^{*}=\arg\,\min_{\mathbf{w}}\left[\int_{0}^{L}H_{1}\left(\mathbf{y},\mathbf{w},\boldsymbol{\lambda},x\right)dx\right]$$

Similarly, if $\mathbf{w}=\mathbf{w}\left(x\right)$, then Eq. (44) is replaced by

$$\mathbf{w}^{*}=\arg\,\min_{\mathbf{w}}\left[\int_{0}^{T}H_{1}\left(\mathbf{y},\mathbf{w},\boldsymbol{\lambda},x\right)dt\right]$$

Equations (44) and (45) are the necessary conditions for optimal controls of the distributed and lumped-parameter systems (S) and (P) regardless of whether or not the controls are bounded. Thus, they are stronger statements than the necessary conditions for optimality obtained from Eqs. (28) and (29). Clearly, if the controls are unbounded, the minimum principle is reduced to the necessary conditions obtained from the variational principle.

Linear-Quadratic Control

Consider a linear-quadratic cost functional of tracking type

$$J=\frac{1}{2}\int_{0}^{T}\int_{0}^{L}\left(\mathbf{y}-\mathbf{y}_{d}\right)^{T}\mathbf{P}\left(\mathbf{y}-\mathbf{y}_{d}\right)dxdt+\frac{1}{2}\int_{0}^{T}\left(\mathbf{w}^{T}\mathbf{Q}\mathbf{w}+\mathbf{v}^{T}\mathbf{R}\mathbf{v}\right)dt \tag{46}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are positive semidefinite matrices, $\mathbf{R}$ is a positive definite matrix, and $\mathbf{y}_d$ is the steady state solution to the following linear hyperbolic PDE constraint

$$\mathbf{y}_{t}+\mathbf{A}\mathbf{y}_{x}+\mathbf{C}\mathbf{y}=\mathbf{0} \tag{47}$$

where $\mathbf{A}$ and $\mathbf{C}$ are constant coefficient matrices. Equation (47) is subject to an initial condition

$$\mathbf{y}\left(x,0\right)=\mathbf{0}$$

Further, it is assumed that the PDE system only has forward wave propagation speeds, i.e., the matrix $\mathbf{A}$ only admits positive-valued eigenvalues, to be consistent with the following boundary condition

$$\mathbf{y}(0,t) = \mathbf{F}\mathbf{u}(t)$$

where $\mathbf{u}(t)$ is the boundary control variable affecting the PDE system, which is further subject to an ODE system dynamics

$$\dot{\mathbf{u}} = \mathbf{G}\mathbf{u} + \mathbf{B}\mathbf{y}(0,t) + \mathbf{H}\mathbf{v}$$

To implement a tracking control, we apply a standard control technique of augmenting the system dynamics with an integral compensator to reduce the steady state error, resulting in the following equation

$$\dot{\mathbf{w}} = \mathbf{G}_w\mathbf{w} + \mathbf{H}_w\mathbf{v} + \mathbf{L}_w\mathbf{u}_d \Leftrightarrow \left\{\begin{array}{c}\dot{\mathbf{e}}\\ \dot{\mathbf{z}}\end{array}\right\} = \left[\begin{array}{cc}\mathbf{G}+\mathbf{B}\mathbf{F} & \mathbf{0}\\ \mathbf{I} & \mathbf{0}\end{array}\right]\left\{\begin{array}{c}\mathbf{e}\\ \mathbf{z}\end{array}\right\} + \left\{\begin{array}{c}\mathbf{H}\\ \mathbf{0}\end{array}\right\}\mathbf{v} + \left\{\begin{array}{c}\mathbf{G}+\mathbf{B}\mathbf{F}\\ \mathbf{0}\end{array}\right\}\mathbf{u}_d \quad (48)$$

where $\mathbf{u}_d$ is a constant setpoint to achieve $\mathbf{y}_d$ so that $\mathbf{y}_d(0) = \mathbf{F}\mathbf{u}_d$, $\mathbf{z}$ is the integral of the error term $\mathbf{e} = \mathbf{u} - \mathbf{u}_d$, and $\mathbf{v}$ is an unbounded control.

The necessary conditions yield the adjoint PDE

$$\boldsymbol{\lambda}_t + \mathbf{A}^T\boldsymbol{\lambda}_x - \mathbf{C}^T\boldsymbol{\lambda} - \mathbf{P}^T(\mathbf{y} - \mathbf{y}_d) = \mathbf{0} \quad (49)$$

subject to the boundary condition

$$\mathbf{A}^T\boldsymbol{\lambda}(L,t) = \mathbf{0}$$

with the adjoint ODE

$$\dot{\boldsymbol{\mu}} + \mathbf{Q}^T\mathbf{w} + \mathbf{G}_w^T\boldsymbol{\mu} - \left\{\mathbf{F}^T\mathbf{A}^T\boldsymbol{\lambda}(0,t) \quad \mathbf{0}\right\}^T = \mathbf{0} \tag{50}$$

and a control

$$\mathbf{v} = -\mathbf{R}^{-T}\mathbf{H}_w^T\boldsymbol{\mu}$$

Using the Laplace transform and characteristic transformation, the general solutions for the PDEs are found to be

$$\mathbf{y}(x,t) = \mathcal{L}^{-1}\Big\{\boldsymbol{\Phi}e^{-x\boldsymbol{\Lambda}^{-1}\left(s\mathbf{I}+\boldsymbol{\Phi}^{-1}\mathbf{C}\boldsymbol{\Phi}\right)}\boldsymbol{\Phi}^{-1}\mathcal{L}\big\{\mathbf{y}(0,t)\big\}\Big\}$$

$$\boldsymbol{\lambda}(x,t) = \mathcal{L}^{-1}\left\{\boldsymbol{\Phi}^{-T}\left[e^{(L-x)\boldsymbol{\Lambda}^{-1}\left(s\mathbf{I}-\boldsymbol{\Phi}^T\mathbf{C}^T\boldsymbol{\Phi}^{-T}\right)}\boldsymbol{\gamma}(L,s) - \boldsymbol{\gamma}(x,s)\right]\right\}$$

where

$$\boldsymbol{\gamma}(x,s) = \left(\boldsymbol{\Phi}^{-1}\mathbf{C}\boldsymbol{\Phi} + \boldsymbol{\Phi}^T\mathbf{C}^T\boldsymbol{\Phi}^{-T}\right)^{-1}\boldsymbol{\Phi}^T\mathbf{P}^T\Big[\mathcal{L}\left\{\mathbf{y}(x,t)\right\} - \mathbf{y}_d(x)\Big]$$

Now if $\boldsymbol{\Phi}^{-1}\mathbf{C}\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^T\mathbf{P}^T\boldsymbol{\Phi}$ are diagonal matrices, then the matrix exponential terms are separable and the solutions may be simplified as

$$\mathbf{y}(x,t) = \boldsymbol{\Phi}e^{-x\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi}^{-1}\mathbf{C}\boldsymbol{\Phi}}\mathbf{f}\left(t - x\boldsymbol{\Lambda}^{-1}\right) \tag{51}$$

$$\boldsymbol{\lambda}(x,t) = \frac{1}{2}\boldsymbol{\Phi}^{-T}\left[e^{-2(L-x)\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi}^{-1}\mathbf{C}\boldsymbol{\Phi}} - \mathbf{I}\right]\boldsymbol{\Phi}^T\mathbf{C}^{-1}\mathbf{P}^T\Big[\mathbf{y}(x,t) - \mathbf{y}_d(x)\Big] \tag{52}$$

where $\mathbf{f}(t) = \boldsymbol{\Phi}^{-1}\mathbf{y}(0,t)$ and

$$\mathbf{f}\left(t - x\boldsymbol{\Lambda}^{-1}\right) = \left\{\begin{array}{cccc} f_1\left(t - x\Lambda_{11}{}^{-1}\right) & f_2\left(t - x\Lambda_{22}{}^{-1}\right) & \ldots & f_n\left(t - x\Lambda_{nn}{}^{-1}\right)\end{array}\right\}^T$$

Thus, the solutions to the PDE system are a superposition of wave propagation solutions corresponding to individual eigenvalues of the coefficient matrix $\mathbf{A}$.

We now consider a special case when the distributed state vector $\mathbf{y}(x,t)$ has a faster time constant than the boundary control state vector $\mathbf{u}(t)$. This is a multiple time-scale problem which can be analyzed by the singular perturbation method. For the present, we will only consider a reduced problem at some time beyond the boundary layer of the solutions where the reduced solution approaches asymptotically to the actual solution. The reduced problem then becomes

$$\varepsilon \begin{Bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{Bmatrix}_t + \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^T \end{bmatrix} \begin{Bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{Bmatrix}_x + \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ -\mathbf{P}^T & -\mathbf{C}^T \end{bmatrix} \begin{Bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ -\mathbf{P}^T \mathbf{y}_d \end{Bmatrix}$$

We consider an approximate solution of the type

$$\boldsymbol{\lambda}(x,t) = \mathbf{A}^{-T}\mathbf{S}(x)\Big[\mathbf{y}(x,t) - \mathbf{y}_d(x)\Big] \tag{53}$$

Substituting into the reduced partial differential equations and letting $\varepsilon \to 0$ yield the following matrix Lyapunov differential equation

$$\frac{d\mathbf{S}}{dx} - \mathbf{S}\mathbf{A}^{-1}\mathbf{C} - \mathbf{C}^T\mathbf{A}^{-T}\mathbf{S} - \mathbf{P}^T = \mathbf{0} \tag{54}$$

subject to a boundary condition

$$\mathbf{S}(L) = \mathbf{0}$$

It may be verified that the solution to Eqs. (53) and (54) yield the same result as Eq. (52).

The system of ODEs now becomes

$$\begin{Bmatrix} \dot{\mathbf{w}} \\ \dot{\boldsymbol{\mu}} \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_w & -\mathbf{H}_w\mathbf{R}^{-T}\mathbf{H}_w^T \\ -\mathbf{Q}_w^T & -\mathbf{G}_w^T \end{bmatrix} \begin{Bmatrix} \mathbf{w} \\ \boldsymbol{\mu} \end{Bmatrix} + \begin{Bmatrix} \mathbf{L}_w\mathbf{u}_d \\ \mathbf{0} \end{Bmatrix}$$

where

$$\mathbf{Q}_w = \mathbf{Q} - \begin{bmatrix} \mathbf{F}^T\mathbf{S}(0)\,\mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

The optimal control is found to be

$$\mathbf{v} = -\mathbf{R}^{-T}\mathbf{H}_w^T\left(\mathbf{W}\mathbf{w} + \mathbf{K}\mathbf{u}_d\right) \tag{55}$$

where $\mathbf{W}$ is a solution to the following matrix Riccati differential equation

$$\dot{\mathbf{W}} - \mathbf{W}\mathbf{H}_w\mathbf{R}^{-T}\mathbf{H}_w^T\mathbf{W} + \mathbf{W}\mathbf{G}_w + \mathbf{G}_w^T\mathbf{W} + \mathbf{Q}_w^T = \mathbf{0} \tag{56}$$

and $\mathbf{K}$ is the solution to the following matrix differential equation

$$\dot{\mathbf{K}} + \left(\mathbf{G}_w^T - \mathbf{W}\mathbf{H}_w\mathbf{R}^{-T}\mathbf{H}_w^T\right)\mathbf{K} + \mathbf{W}\mathbf{L}_w = \mathbf{0} \tag{57}$$

The final-time conditions for both equations are

$$\mathbf{W}\left(T\right) = \mathbf{0}$$

$$\mathbf{K}\left(T\right) = \mathbf{0}$$

The boundary control vector $\mathbf{u}\left(t\right)$ can now be obtained from

$$\dot{\mathbf{u}} = \left(\mathbf{G} - \mathbf{H}\mathbf{R}^{-T}\mathbf{H}^T\mathbf{W}_{11}\right)\mathbf{u} - \mathbf{H}\mathbf{R}^{-T}\mathbf{H}^T\mathbf{W}_{12}\mathbf{z} - \mathbf{H}\mathbf{R}^{-T}\mathbf{H}^T\left(\mathbf{K}_1 - \mathbf{W}_{11}\right)\mathbf{u}_d \tag{58}$$

As an example, we consider a linearized flow about an equilibrium point. Since the mass flow and total temperature are weak functions in x for an adiabatic flow, we may obtain an approximate PDE for the total pressure by ignoring the spatial partial derivatives of the mass flow and total temperature in Eq. (1) so that

$$\frac{\partial p_0}{\partial t} + \bar{u}\left[1 - (k-1)\frac{\bar{T}}{\bar{T}_0}\right]\frac{\partial p_0}{\partial x} + \frac{k\bar{u}^3}{2\bar{c}^2}\frac{\bar{f}}{D}\left[1 - (k-1)\frac{\bar{T}}{\bar{T}_0}\right]p_0 \approx 0 \tag{59}$$

where the overbar denotes the steady state values. Equation (59) is a scalar, linear advection equation of the form

$$y_t + ay_x + cy = 0$$

Suppose the total pressure at the duct inlet is subject to a time-varying boundary condition described by a control

$$y\left(0, t\right) = u\left(t\right)$$

where $u\left(t\right)$ is in turn prescribed by a first-order ODE control action

$$\dot{u} = gu + hv$$

The objective is to find $v\left(t\right)$ that optimally tracks a unit step input $u_d = 1$ to the system initially at rest so that $y\left(x, T\right) \to y_d\left(x\right) = u_d e^{-\frac{cx}{a}}$ in a linear-quadratic sense

$$J = \frac{1}{2}\int_0^L \int_0^T p\left(y - y_d\right)^2 dxdt + \frac{1}{2}\int_0^T \left(\mathbf{w}^T\mathbf{Q}\mathbf{w} + rv^2\right) dt$$

We introduce $\mathbf{w} = \left\{ \begin{array}{cc} u - u_d & z = \int_0^t \left(u - u_d\right) dt \end{array} \right\}^T$ as an integrally compensated augmented state vector with a dynamics

$$\dot{\mathbf{w}} = \mathbf{G}\mathbf{w} + \mathbf{H}v + \mathbf{L}u_d$$

where

$$\mathbf{G} = \begin{bmatrix} g & 0 \\ 1 & 0 \end{bmatrix}, \ \mathbf{H} = \left\{ \begin{array}{c} h \\ 0 \end{array} \right\}, \ \mathbf{L} = \left\{ \begin{array}{c} g \\ 0 \end{array} \right\}$$

Proceeding to find the adjoint solution from Eq. (54), the Lyapunov differential equation becomes

$$\frac{dS}{dx} - \frac{2c}{a}S - p = 0$$

whose solution with the boundary condition $S(L) = 0$ is

$$S(x) = -\frac{pa}{2c}\left[1 - e^{-\frac{2a(L-x)}{c}}\right]$$

The matrix $\mathbf{Q}$ is then augmented with $S(0) = -\frac{pa}{2c}\left(1 - e^{-\frac{2aL}{c}}\right)$ in the first diagonal element. The solution for the adjoint partial differential equation then becomes

$$\lambda(x,t) = \frac{1}{a}S(x)\left[y(x,t) - y_d(x)\right]$$

where

$$y(x,t) = e^{-\frac{cx}{a}} f\left(t - \frac{x}{a}\right) \tag{60}$$

Eq. (60) is a boundary-value problem for which the wave propagation function $f(t) = u(t)$ is to be determined from the boundary control solution of $u(t)$ in Eq. (58)

$$\dot{u} = \left(g - \frac{h^2 W_{11}}{r}\right)u - \frac{h^2 W_{12}}{r}z - \frac{h^2 K}{r} \tag{61}$$

where $\mathbf{W}$ and K are the steady-state solutions of Eqs. (56) and (57). Equation (61) may be written as

$$\ddot{u} - \left(g - \frac{h^2 W_{11}}{r}\right)\dot{u} + \frac{h^2 W_{12}}{r}(u - u_d) = 0$$

with initial conditions $u(0) = 0$ and $\dot{u}(0) = -\frac{h^2 K}{r}$. The solution of $u(t)$ is then obtained to be

$$u(t) = u_d\left\{1 - e^{-\sigma t}\left[\cos\omega t + \left(\frac{h^2 K}{r\omega u_d} - \frac{\sigma}{\omega}\right)\sin\omega t\right]\right\} \tag{62}$$

where $\sigma = \frac{1}{2}\left(\frac{h^2 W_{11}}{r} - g\right)$ and $\omega = \sqrt{\frac{h^2 W_{11}}{r} - \frac{1}{4}\left(\frac{h^2 W_{11}}{r} - g\right)^2}$.

The optimal solution of $y\left(x,t\right)$ is now determined to be

$$y\left(x,t\right)=\begin{cases} 0 & t\leq\frac{x}{a} \\ e^{-\frac{cx}{a}}u_{d}\left\{1-e^{-\sigma\left(t-\frac{x}{a}\right)}\left[\cos\omega\left(t-\frac{x}{a}\right)+\left(\frac{h^{2}K}{r\omega u_{d}}-\frac{\sigma}{\omega}\right)\sin\omega\left(t-\frac{x}{a}\right)\right]\right\} & t>\frac{x}{a}\end{cases}$$

The solution surfaces of the PDE system are plotted on Figs. 3 and 4.

The piecewise optimal solution of $y\left(x,t\right)$ reflects the wave propagation nature of hyperbolic partial differential equations. The system is initially at rest and remains so until the initial disturbance due to the unit-step input at the boundary has propagated through the system. The PDE system eventually would reach a new equilibrium at $y_{d}\left(x\right)$. Every point $x>0$ is subject to a time delay effect of $t_{d}=x/a$. In fact, the PDE may be written in a semi-discretized form as a time-delay ODE

$$\dot{y}\left(t\right)=-\left(\frac{a}{\Delta x}+c\right)y\left(t\right)+\frac{a}{\Delta x}y\left(t-\Delta t_{d}\right)$$

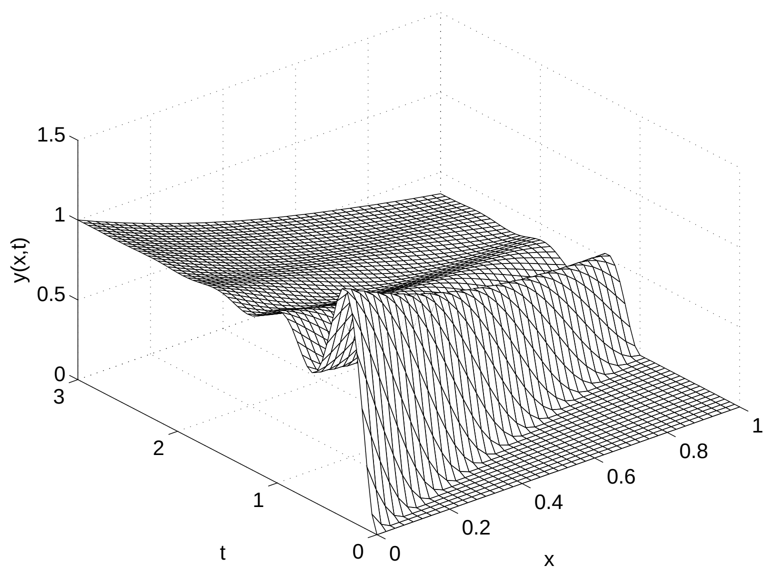

Fig. 3 - Extremal Boundary Control Solution Surface of $y\left(x,t\right)$

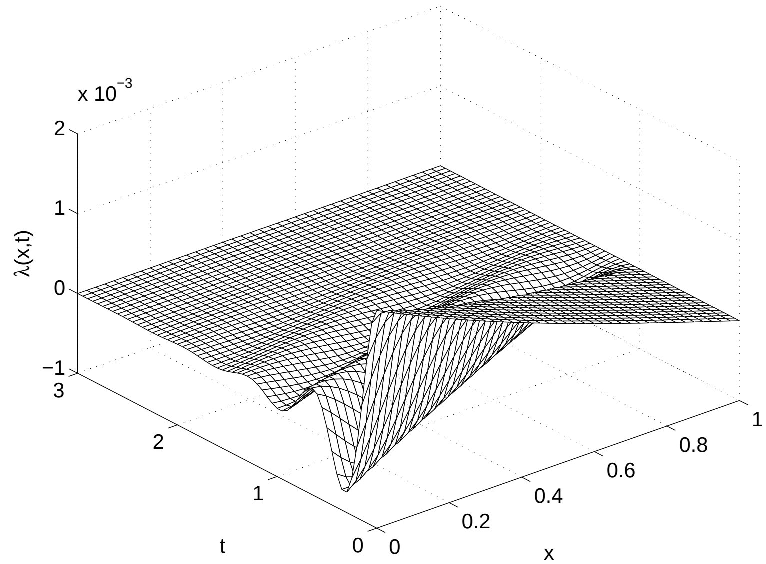

Fig. 4 - Adjoint Solution Surface of $\lambda\,(x,t)$

The foregoing problem is an example of a simple hyperbolic system with wave propagation in only the downstream direction. In such a case, the boundary conditions must provide information from the upstream end of the solution domain. More often than not, waves in 1-D systems such as subsonic flow or elastic solids generally propagate in both directions. The boundary conditions therefore must be specified accordingly at both ends of the solution domain in order to establish well-posed boundary conditions. If n is the number of positive eigenvalues of the matrix $\mathbf{A}$, then n independent boundary conditions at the upstream end must be specified, while the remaining boundary conditions must be imposed at the downstream end. Equation (10) provides general boundary conditions for handling a hyperbolic system that involves wave propagation in both directions. This type of boundary condition is generally classified as non-homogeneous periodic boundary conditions and typically is more difficult to handle, since information at both ends of the solution domain influences the boundary control solution at each instance in time. In this example problem, the adjoint PDE and ODE systems are uncoupled, but generally for a periodic boundary value problem, the PDE and ODE systems are coupled together through the boundary condition, thus further complicating optimal control solutions.

Applications to Wind Tunnel Flow Control

Optimal flow control of a wind tunnel can be synthesized using the theory developed in the present study. In future wind tunnel flow control studies, two types of optimal control problems are to be considered: trajectory optimization and adaptive disturbance feedforward control. In determining an optimal trajectory for transferring a flow condition in a wind tunnel from one equilibrium to another, the optimality conditions involving the two PDE systems and two ODE systems must be solved concurrently. This is a two point boundary value problem whose solutions may be obtained by various optimization techniques such as the conjugate gradient method. Prior to forming a numerical solution, the system and adjoint PDEs must be discretized by various numerical techniques. Because of the time evolution nature of the solutions, the stability of a numerical discretization must be considered. The Courant-Friedrichs-Lewy (CFL) condition generally must be satisfied [13].

Another problem of interest for a wind tunnel control application is adaptive disturbance feedforward control. The problem may be formulated as a linear PDE system with a disturbance input

$$\mathbf{y}_t + \mathbf{A}\mathbf{y}_x + \mathbf{B}\mathbf{y} + \mathbf{C}w = \mathbf{0}$$

where w is the disturbance which physically represents the test model drag coefficient. Typically, during a wind tunnel test, the model is actuated through a series of angles of attack. Changes in the flow around the test model cause the flow condition in the wind tunnel to meander from a setpoint. Presently, to minimize this flow deviation, the test model has to be paused in between changes in the pitch angle so that the flow condition can be regulated. It would be desirable to improve the flow control strategy by allowing the test model to be actuated continuously while the flow condition would be maintained near its setpoint.

The proposed control strategy would employ an adaptive neural network learning algorithm to predict the model-induced loss factor. The control variables would then be calculated based on a receding horizon optimal control approach to predict a feedforward control in coordination with the model actuation to maintain the flow condition as close to the set point as possible. A receding horizon optimal control deals with a control during a short time horizon throughout which the time-varying disturbance may be assumed constant. A typical cost functional may be as follows

$$J = \frac{1}{2}\int_t^{t+T}\int_0^L \mathbf{y}^T\mathbf{P}\mathbf{y}\,dx\,d\tau + \frac{1}{2}\int_t^{t+T}\left(\mathbf{u}^T\mathbf{Q}\mathbf{u} + \mathbf{v}^T\mathbf{R}\mathbf{v}\right)d\tau$$

Concluding Remarks

This paper presents some recent results in optimality conditions for boundary control of a distributed system governed by first order, quasilinear hyperbolic partial differential equations in the presence of a lumped-parameter system at the boundary defined by ordinary differential equations. The formulation in terms of Hamiltonian functions and differential operators provides some similarity in optimality conditions to those of ordinary differential equations. A linear, time-invariant system was provided as an example to demonstrate the application of the theory in deriving a boundary control feedback law. A proposed solution was given in a form of a quasi-steady state control obtained by solving a matrix Lyapunov differential equation in space and a matrix Riccati differential equation in time successively. Wind tunnel control applications based on the present theory were discussed for future work.

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